

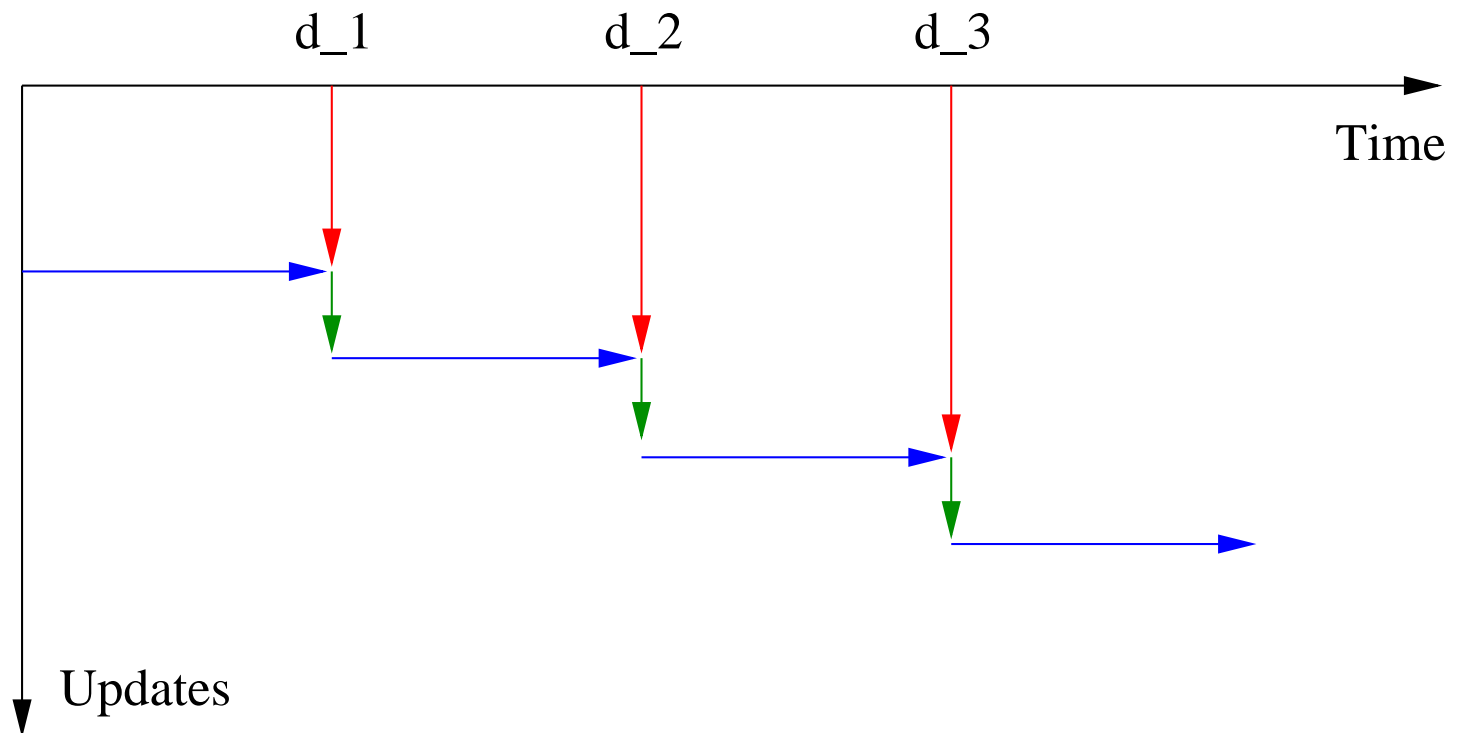
Kalman Filtering

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Kalman Filter

- “Approximate” solution of inverse problem
- Sequentially updates model state and uncertainty
- Variance minimizing update step
- Estimate improves and uncertainty reduces at each update



Analysis scheme

Given at time t :

- First guess estimate, $\psi^f(\mathbf{x})$.
- Error covariance of first guess, $C_{\psi\psi}^f(\mathbf{x}_1, \mathbf{x}_2)$.
- Vector of measurements, $\mathcal{M}\psi^t = \mathbf{d} + \epsilon$.
- Error covariance for measurements, $C_{\epsilon\epsilon}$.

and assume Gaussian statistics!

$$\begin{aligned} \mathcal{J}[\psi] = & \iint_{\mathcal{D}} (\psi - \psi^f) (C_{\psi\psi}^f)^{-1} (\psi - \psi^f)^T d\mathbf{x}_1 d\mathbf{x}_2 \\ & + (\mathcal{M}\psi - \mathbf{d}) C_{\epsilon\epsilon}^{-1} (\mathcal{M}\psi - \mathbf{d})^T \end{aligned}$$

Analysis scheme

An optimal estimator is then

- Analysis update

$$\psi^a = \psi^f + C_{\psi\psi}^f \mathcal{M}^T (\mathcal{M} C_{\psi\psi}^f \mathcal{M}^T + C_{\epsilon\epsilon})^{-1} (d - \mathcal{M} \psi^f)$$

- Error covariance update

$$C_{\psi\psi}^a = C_{\psi\psi}^f - C_{\psi\psi}^f \mathcal{M}^T (\mathcal{M} C_{\psi\psi}^f \mathcal{M}^T + C_{\epsilon\epsilon})^{-1} \mathcal{M} C_{\psi\psi}^f.$$

Analysis scheme

Properties of the estimator

- Linear, unbiased and variance minimizing.
- MLH estimate for Gaussian statistics.
- Dimension of problem equals number of measurements.
- Can be derived from statistical or variational formulation.
- Consistent with Bayes.

Kalman Filter (KF)

- Includes time dimension.
- Idea is to:
 1. predict $\psi^f(\mathbf{x})$ and $C_{\psi\psi}^f(\mathbf{x}_1, \mathbf{x}_2)$,
 2. use analysis scheme to update these whenever measurements are available.
- Sequential filtering method!
 - Information from measurements carried forward in time.

KF: Error evolution

Derivation for linear scalar model

- Evolution of **true** state

$$\psi_k^t = F\psi_{k-1}^t + q_{k-1}$$

- Our model is

$$\psi_k^f = F\psi_{k-1}^a,$$

with **forecast** f and **analysis** a .

- Difference is

$$\psi_k^t - \psi_k^f = F(\psi_{k-1}^t - \psi_{k-1}^a) + q_{k-1}$$

KF: Error covariance evolution

Error covariance equation

$$\begin{aligned}C_{\psi\psi}^f(t_k) &= \overline{(\psi_k^t - \psi_k^f)^2} \\&= F^2 \overline{(\psi_{k-1}^t - \psi_{k-1}^a)^2} + \overline{q_{k-1}^2} + 2F \overline{(\psi_{k-1}^t - \psi_{k-1}^a)q_{k-1}} \\&= F^2 C_{\psi\psi}^a(t_{k-1}) + C_{qq}(t_{k-1}).\end{aligned}$$

with

- $C_{\psi\psi}^a(t_{k-1}) = \overline{(\psi_{k-1}^t - \psi_{k-1}^a)^2}$,
- $C_{qq}(t_{k-1}) = \overline{(q_{k-1})^2}$,
- model errors uncorrelated with state error.

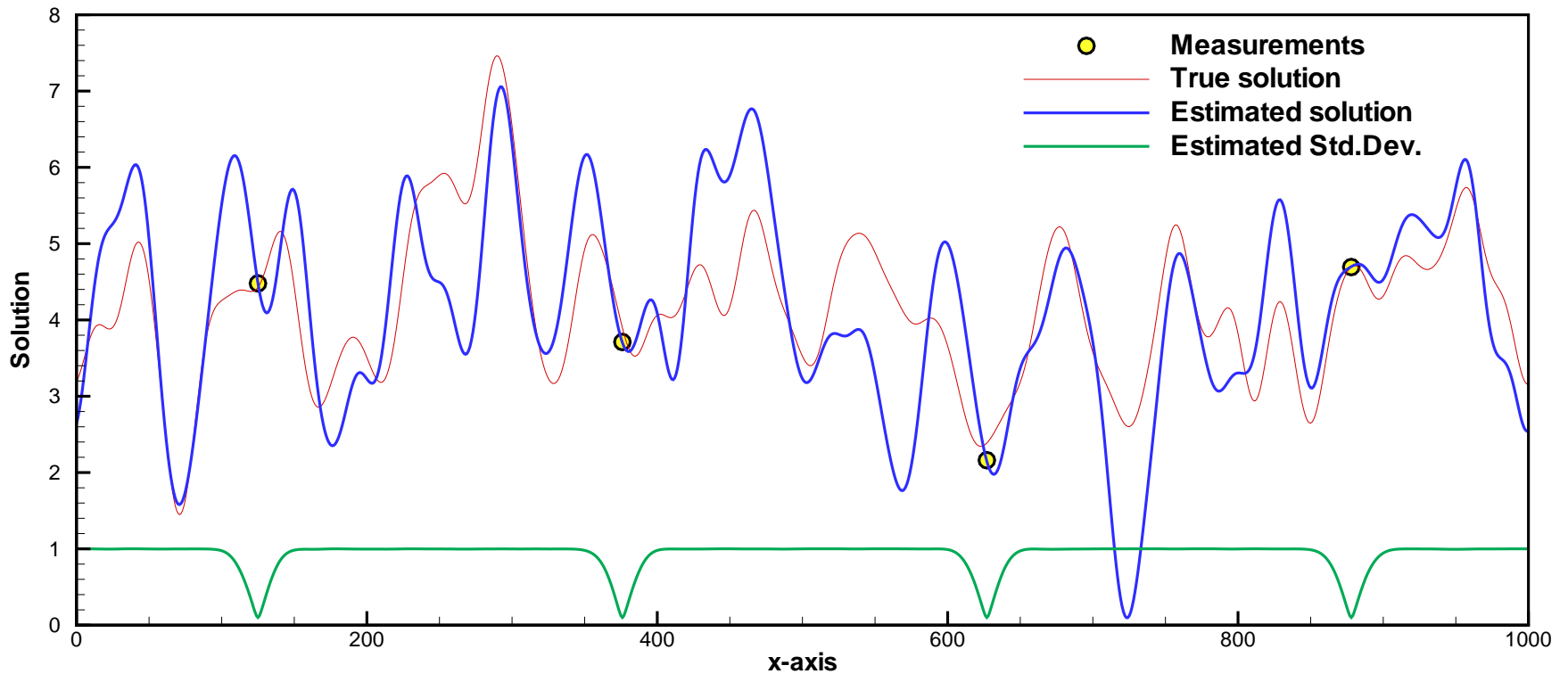
Kalman Filter Example

Model

- Linear advection equation
- Periodic domain
- $u = \Delta t = \Delta x = 1.0$
- Random reference solution.
- First guess is reference plus random perturbation.
- Initial variance is 1.0
- Four measurements every 5 time units.
- Measurement variance is 0.01.
- Cases without and including system noise = 0.0004.

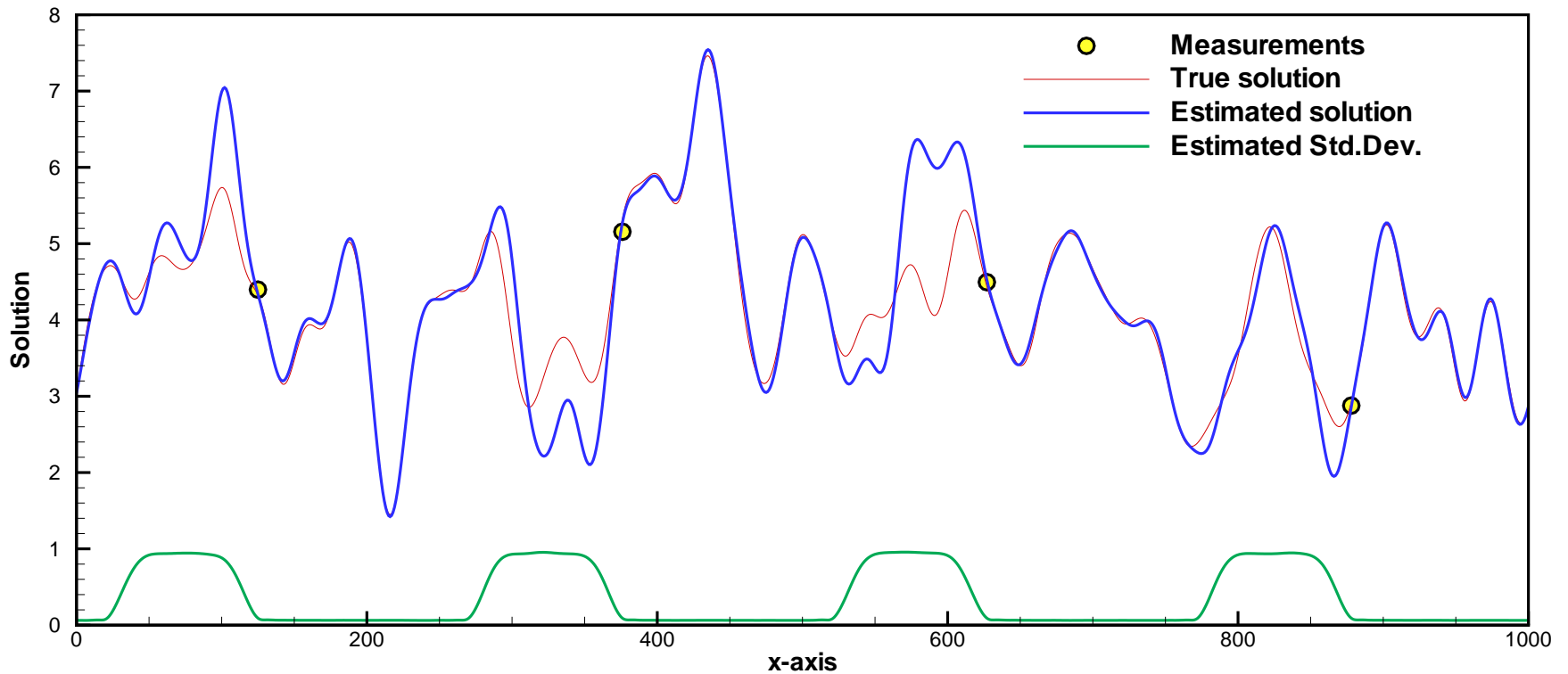
Kalman Filter Example: Case A

Solution after first update $t = 5.0$



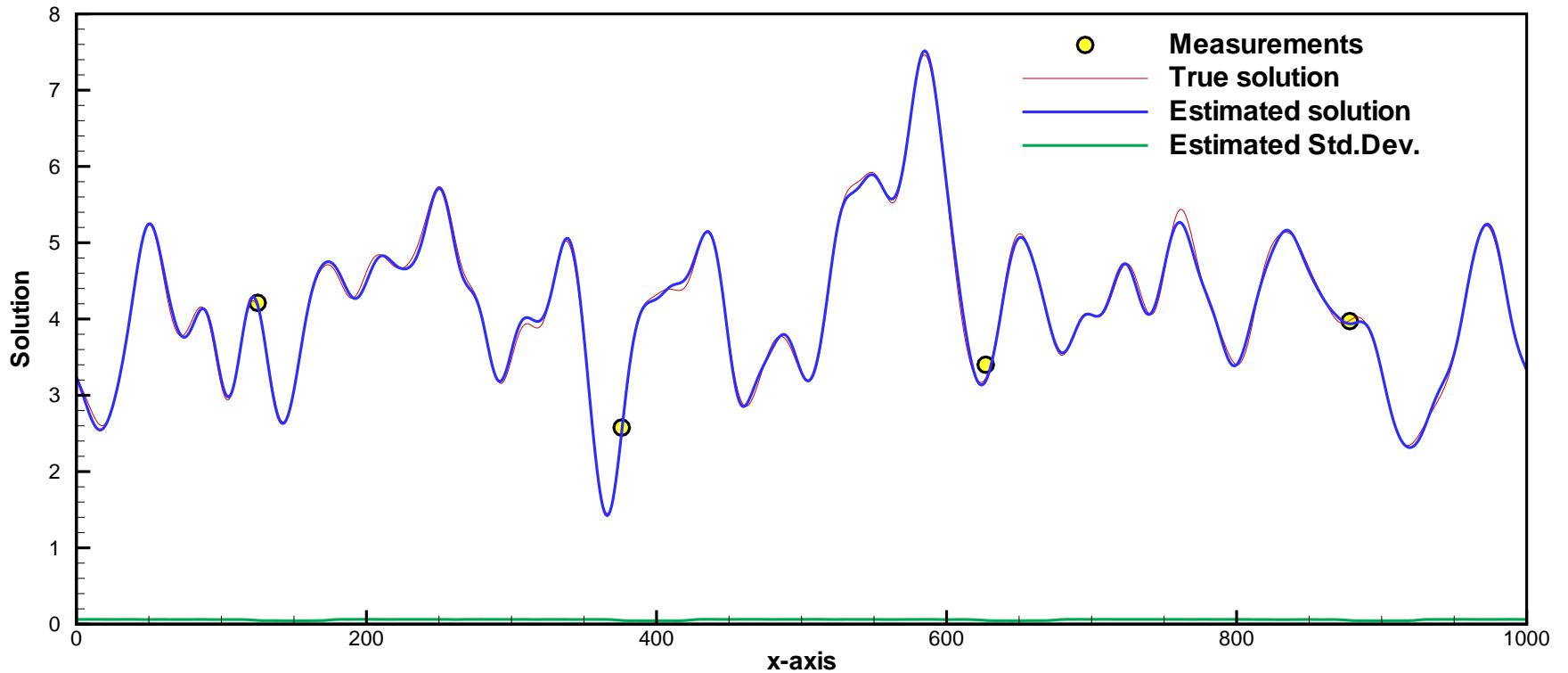
Kalman Filter Example: Case A

Solution after 30 updates $t = 150.0$



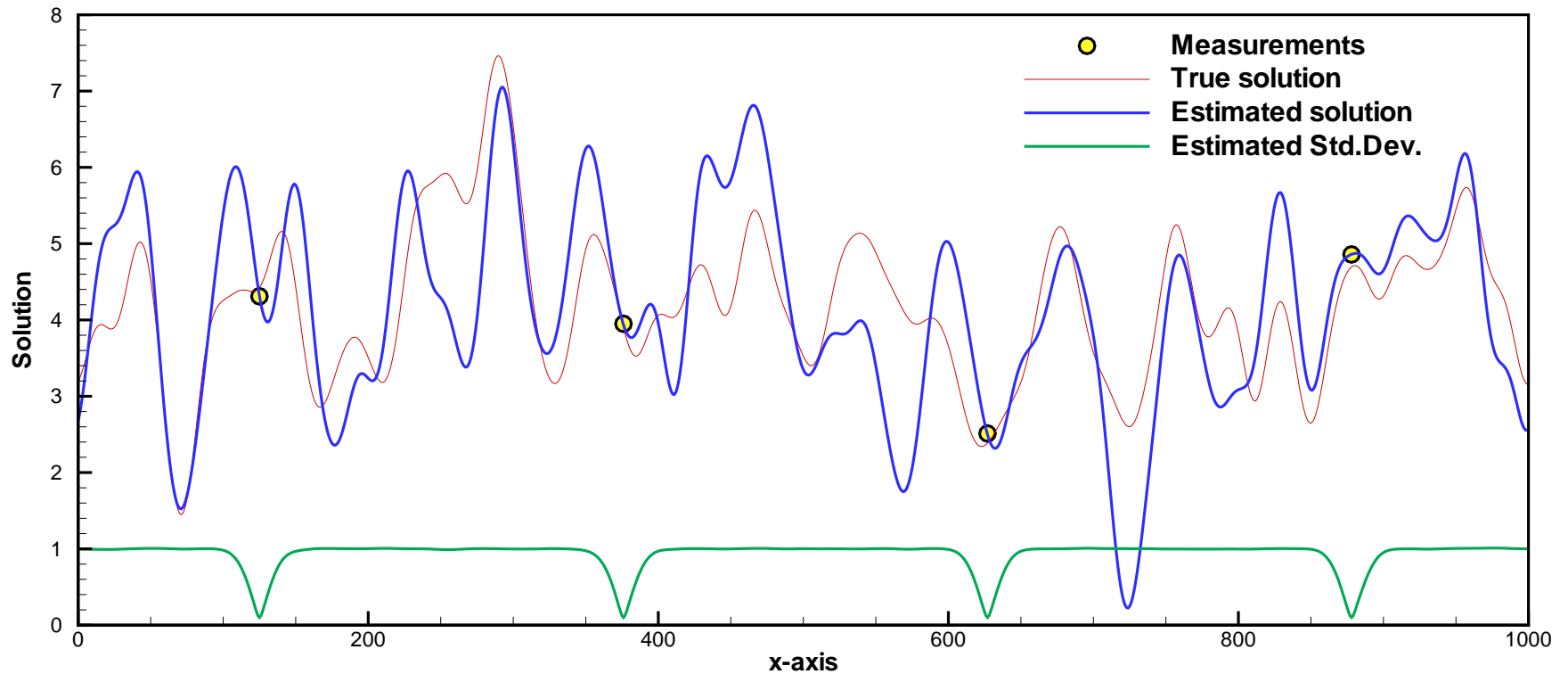
Kalman Filter Example: Case A

Solution after 60 updates $t = 300.0$



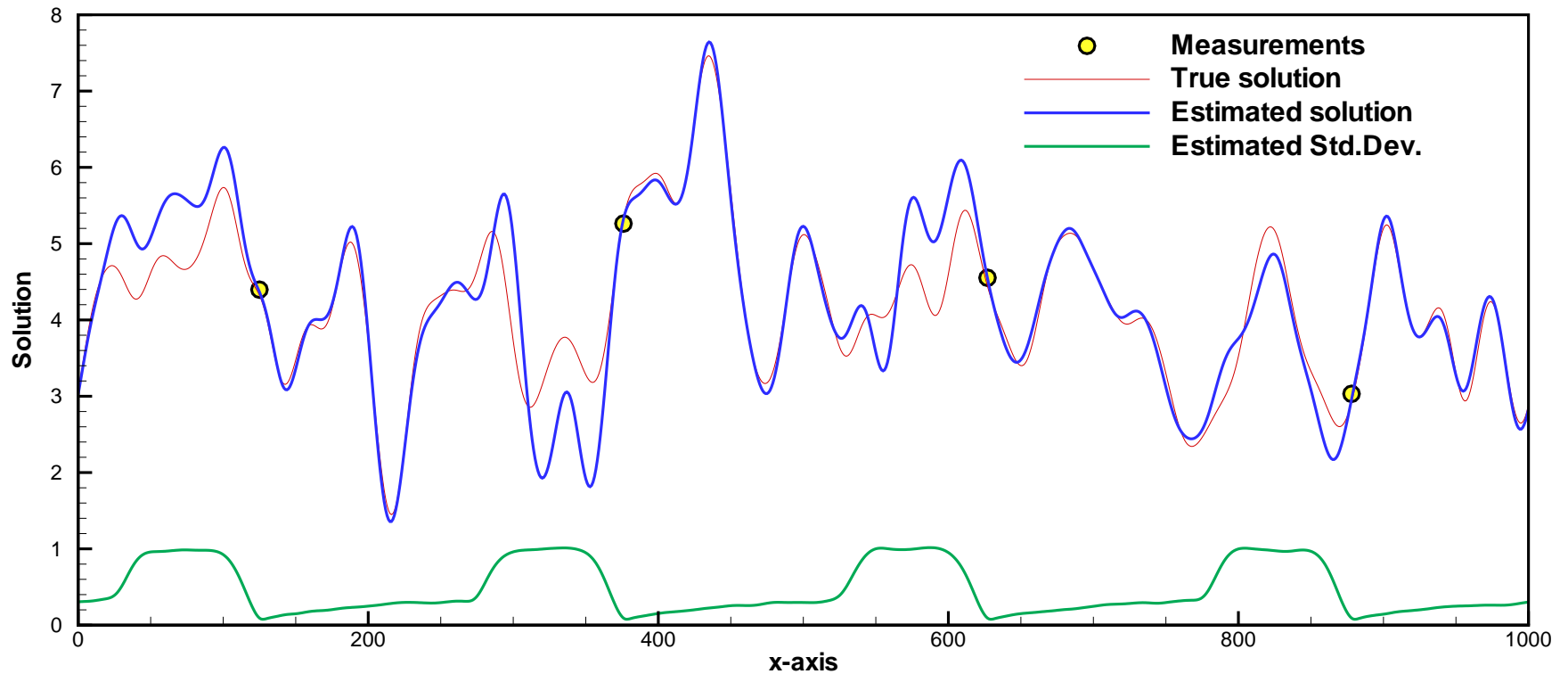
Kalman Filter Example: Case B

Solution after first update $t = 5.0$



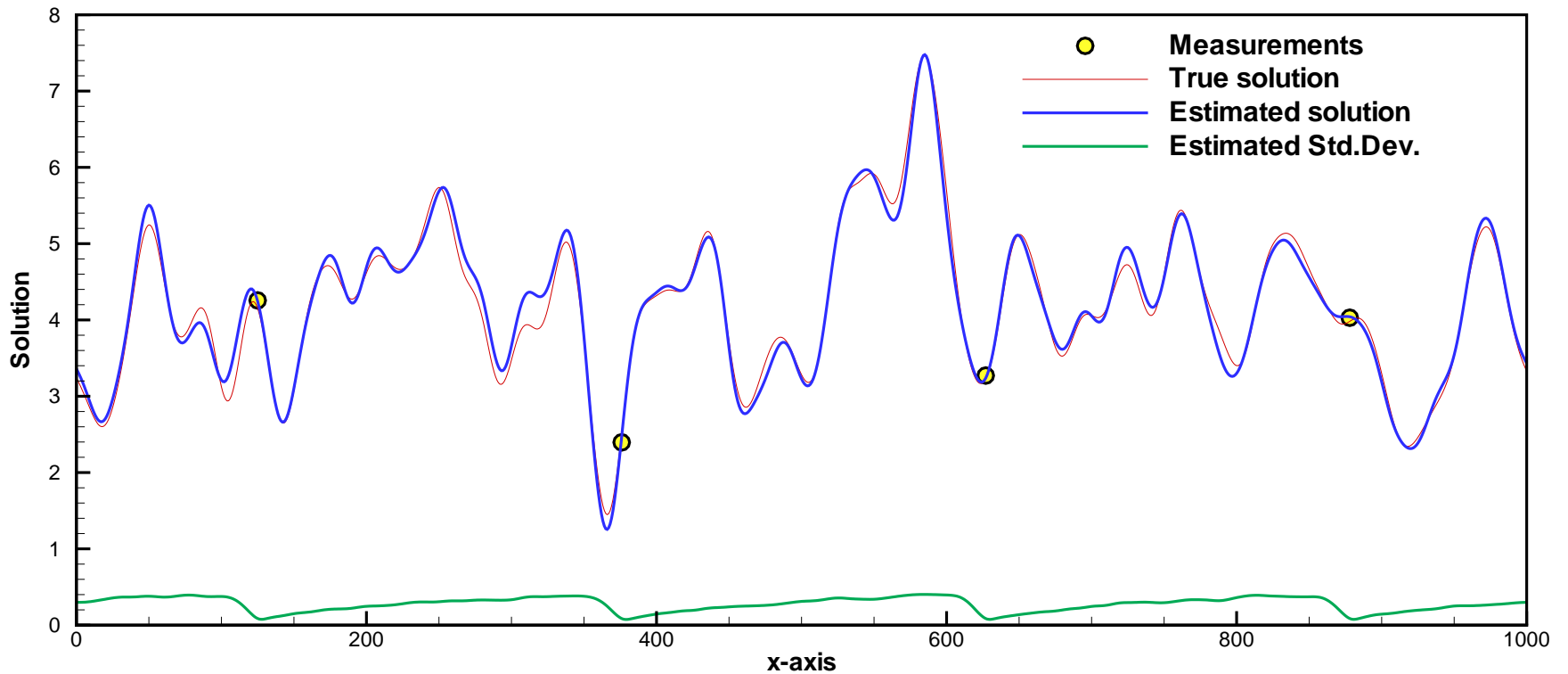
Kalman Filter Example: Case B

Solution after 30 updates $t = 150.0$

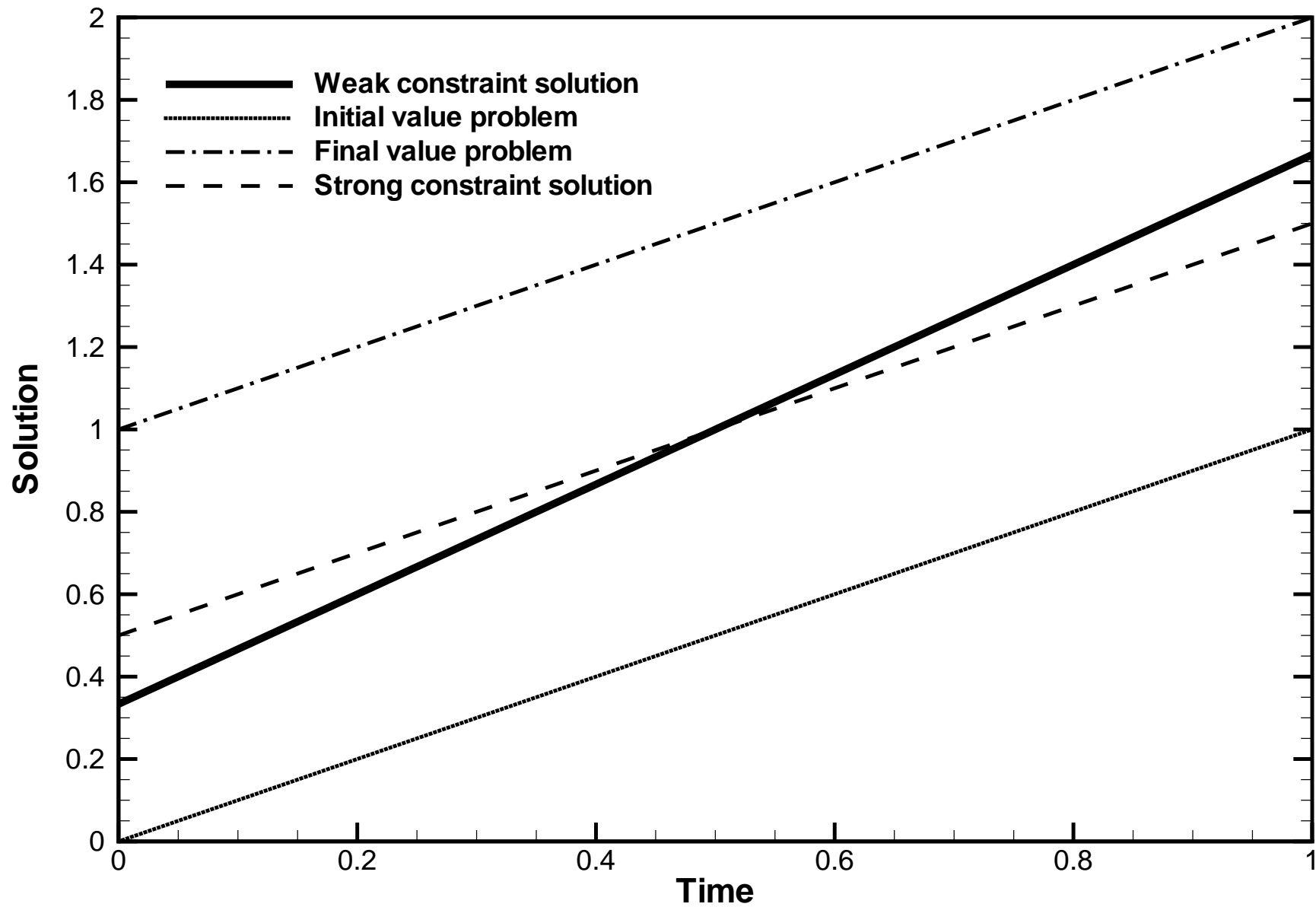


Kalman Filter Example: Case B

Solution after 60 updates $t = 300.0$



Inverse problem revisited



KF solution

What is the KF solution for the linear inverse problem?

- Solve initial value problem until $t = 1$.
- The predicted error variance equals $2C$ at $t = 1$.
- Update at $t = 1$ is

$$\begin{aligned}\psi^a &= \psi^f + \frac{C_{\psi\psi}^f}{C_{\epsilon\epsilon} + C_{\psi\psi}^f} (d - \psi^f) \\ &= 1 + \frac{2C}{C + 2C} (2 - 1) \\ &= 5/3\end{aligned}$$

- KF solution at final time equals weak constraint variational solution.

Nonlinear dynamics

- Derivation of Extended Kalman Filter (EKF)

Nonlinear scalar model

$$\psi_k^t = F(\psi_{k-1}^t) + q_{k-1},$$

$$\psi_k^f = F(\psi_{k-1}^a),$$

$$\psi_k^t - \psi_k^f = F(\psi_{k-1}^t) - F(\psi_{k-1}^a) + q_{k-1}.$$

Use Taylor expansion

$$\begin{aligned} F(\psi_{k-1}^t) &= F(\psi_{k-1}^a) + F'(\psi_{k-1}^a)(\psi_{k-1}^t - \psi_{k-1}^a) \\ &\quad + \frac{1}{2}F''(\psi_{k-1}^a)(\psi_{k-1}^t - \psi_{k-1}^a)^2 + \dots \end{aligned}$$

EKF: Derivation

Difference becomes

$$\begin{aligned}\psi_k^t - \psi_k^f &= F'(\psi_{k-1}^a)(\psi_{k-1}^t - \psi_{k-1}^a) \\ &+ \frac{1}{2}F''(\psi_{k-1}^a)(\psi_{k-1}^t - \psi_{k-1}^a)^2 + \dots + q_{k-1}.\end{aligned}$$

By squaring and taking the expectation we get

$$\begin{aligned}C_{\psi\psi}^f(t_k) &= \overline{(\psi_k^t - \psi_k^f)^2} \\ &= \overline{(\psi_{k-1}^t - \psi_{k-1}^a)^2 (F'(\psi_{k-1}^a))^2} + \frac{1}{2} \overline{(\psi_{k-1}^t - \psi_{k-1}^a)^3 F'(\psi_{k-1}^a) F''(\psi_{k-1}^a)} \\ &+ \frac{1}{4} \overline{(\psi_{k-1}^t - \psi_{k-1}^a)^4 (F''(\psi_{k-1}^a))^2} + \dots + C_{qq}(t_{k-1}).\end{aligned}$$

EKF: Error evolution

Close by discarding high order moments to get

$$\psi_k^f = F(\psi_{k-1}^a),$$

$$C_{\psi\psi}^f(t_k) \simeq C_{\psi\psi}^a(t_{k-1})(F'(\psi_{k-1}^a))^2 + C_{qq}(t_{k-1}),$$

together with standard analysis equations.

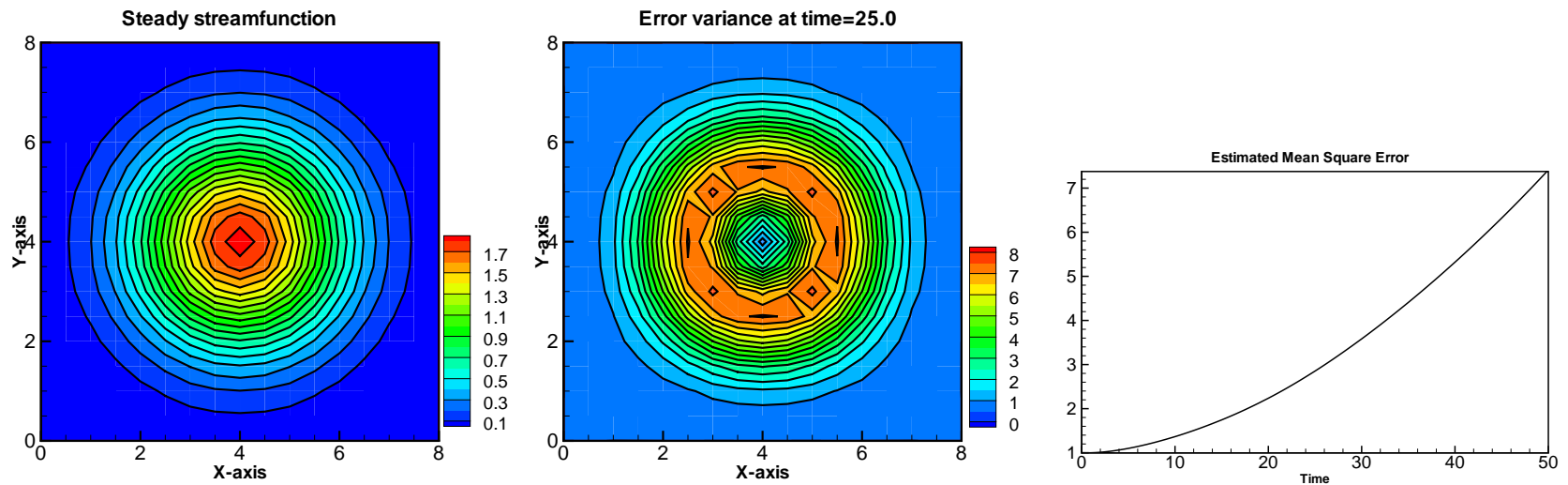
Example of EKF

Nonlinear quasi-geostrophic model

- Steady stream function solution.
- Curved and sheared flow.
- Supports instability.
- Initial variance is 1.0.

Example of EKF

Results (from Evensen 1992):



- Linear closure approximation not valid!
- Leads to linear instability and exponential error growth.

EKF: Summary

- KF is optimal linear filter method!
- EKF applies closure approximation in error covariance equation.
- Does not work for strongly nonlinear models.
 - May lead to linear instabilities in error covariance equation.
 - To simple closure.
- Furthermore, the model equation is not the correct one!

EKF: Equation for the mean

- EKF integrates equation for “central forecast”.
- For nonlinear dynamics the central forecast and mean differs.
- An equation for the mean is easily derived as

$$\overline{\psi}_k = \mathbf{F}(\overline{\psi}_{k-1}) + \frac{1}{2} \mathcal{H}_{k-1} \mathbf{C}_{\psi\psi}(t_{k-1}) + \dots$$

with \mathcal{H} the Hessian of F .

Kalman filtering summary

- Cost of KF/EKF is
 1. storage of $\mathcal{O}(n^2)$ elements,
 2. integration of $2n$ models,
 3. implementation of tangent linear model in EKF,
 4. implementation of Hessian operator and evaluation of $\mathcal{H}_{k-1} \mathbf{C}_{\psi\psi}(t_{k-1})$ if equation for the mean is used.
- Consistency limited by
 1. linear equation for error covariance evolution,
 2. normally the use of the central forecast equation.